THE DESIGN OF DIGITAL FILTERS FOR BIOMEDICAL SIGNAL PROCESSING

PART 3: THE DESIGN OF BUTTERWORTH AND CHEBYCHEV FILTERS

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ABSTRACT
The first two papers in this series reviewed the basic concepts which apply to digital filter theory and presented design techniques based on the z plane pole-zero plot. In this paper these methods are used to develop digital versions of Butterworth and Chebychev filters. The basic theory of both filter types is reviewed and the bilinear transformation is used to derive the z-transforms of the filters from their s-plane continuous time descriptions. Recurrence relationships which may be used to implement filters of various orders are developed. The impulse and frequency responses of the elements are illustrated and examples are given of their application to ECG data.

Keywords: Electric filters, digital filters, signal processing, transform calculus.

INTRODUCTION
The first two parts of the paper considered the properties and design of a number of different types of filter, both in terms of the s domain and the z domain. In all cases the problem of design is essentially the same, that is trying to allow some frequencies to pass through the filter while stopping others. The ideal case therefore is unity gain in the pass band and an instantaneous drop to ‘zero’ in the stop band. For simple filters, for example the simple low-pass case, the characteristic can be represented in very approximate terms by a single s-plane pole. The disadvantage of this type of filter is that in the stop band the characteristic only falls at 6 db per octave, or 20 db per decade. Theoretically this difficulty could be overcome by increasing the order of the pole. However, if this method is used problems arise with regard to the phase characteristics.

The previous parts of the paper have discussed a number of different filter specifications. To reiterate, the essential feature of a filter is that it should stop unwanted frequencies; design therefore normally concentrates on the curve of the modulus \(|G(j\omega)|\) or more usefully the modulus of \(|G(j\omega)|^2\). Various functions are tried until the appropriate one is found. Once the amplitude function is obtained, it is very important to derive the transfer function \(G(s)\) to ensure that the function is realizable. It is also common when designing filters to work with normalized values i.e. with cut-off frequency set at 1 radian per second.

Two famous filter types which have their origin as s-plane designs are the so-called Butterworth and Chebychev filters. Later in this paper we investigate the development of digital versions of these filters, but first we reiterate some comments on digital filters in general.

Because of the widespread use of digital computers and special purpose digital hardware, digital filtering has become an important technique in the area of biomedical signal analysis. The techniques covered in the previous section are essential to the design of digital filters, in particular the use of the z-plane. In addition, digital filtering is an intrinsic part of many computer simulation techniques in...
the study of biomedical systems, and because of the decreasing size and cost of digital hardware it is now feasible that special purpose hardware can be designed for real time digital filtering. Digital filters have several inherent advantages over continuous time hardware filters: first a greater degree of precision can be obtained in a digital realization; second, a greater variety of digital filters can be built since certain realization problems, for example that of negative inductance, do not arise; no special components are needed to realize digital filters with time varying coefficients. Isolation between cascaded digital filter elements is inherent in their design; they do not, therefore, suffer from the problems of input and output impedance compatibility associated with many cascaded analogue designs. The greater simplicity of design which can often be achieved in the digital domain leads to the possibility of more sophisticated types of filter being realized with relative ease.

Digital versions of the well known Butterworth and Chebychev analogue filters will now be considered. We begin by discussing their properties in the Laplace $s$-domain and invoke the bilinear transformation to derive their digital forms. It will be shown that for a particular order of filter a general algorithm can be developed so that the cut-off frequency can be expressed as a parameter in the argument list of a subroutine call. Three filters will be considered in detail; the 2nd order Butterworth, the 4th order Butterworth and the 3rd order Chebychev. For the purpose of comparison, the same physiological signal, a single lead ECG with ectopic beats will be processed by these different filters.

THE BUTTERWORTH APPROXIMATION

Theory

We begin by developing the Butterworth filter, first from the point of view of its basic transfer function, and second by way of an example of the design of a second order filter. A more detailed development of a fourth order version in the $s$-plane then follows. On the basis of the transfer function it is possible to derive a series of transfer functions for the different orders of the Butterworth filter.

It is customary to express the response of such filters in terms of the square of the transfer function. Hence the transfer function is given by:

$$G(s)G(-s) = G(s)^2$$

and to obtain the frequency response $\omega$ is set to zero, hence $s = j\omega$. The functions of interest are $|G(s)|^2$ and $|G(j\omega)|^2$.

Consider the case of a single pole expressed in normalized units:

$$G(s) = \frac{1}{s + 1}$$

hence

$$G(s)G(-s) = \frac{1}{(1 + s)(1 - s)} = \frac{1}{1 - s^2}$$

and the equivalent frequency response becomes:

$$|G(j\omega)|^2 = \frac{1}{1 + \omega^2}$$

The basic expression was modified by Butterworth in order to provide a better approximation for the ideal filter.

For a multi-pole approximation of order $n$:

$$|G(j\omega)|^2 = \frac{1}{(1 + \omega^2)^n} \quad (1)$$

In this standard multi-pole expression, as $n$ increases the break point of the filter (-3db point) remains constant.

The expression developed by Butterworth modifies equation (1) to the form:

$$G(j\omega)^2 = \frac{1}{1 + \omega^{2n}} \quad (2)$$

where $n = 1, 2, 3 \ldots k$ and defines the order of the filter.

The Butterworth approximation is often called the 'maximally flat' approximation. The reason for this alternative name is clear if equivalent frequency responses are obtained from equations (1) and (2). Figure 1 illustrates the frequency responses for 3rd and 6th order Butterworth approximations and their simple pole equivalents. Referring to the figure, it can be seen that in the pass-band the Butterworth approximation remains nearer to unity gain as the frequency scale approaches cut-off (1 rad sec$^{-1}$). In the attenuation band the two filter approximations converge by approximately 2 octaves beyond the cut-off frequency. The improvement obtained from the Butterworth approximation is very evident for the 6th order case. Referring to Figure 1, it is clear that in the pass band the filter has virtually unity gain until just below the cut-off frequency.
frequency and matches the simple pole approximation exactly within one decade.

A general expression for the pole locations of an $n$th order Butterworth filter can be derived from equation (2).

If $s = j\omega$ then $\omega = s/j$

Hence $G(j\omega)^2$ becomes

$$
\frac{1}{1 + (s/j)^{2n}}
$$

and the poles of the filter are therefore given by:

$$(s/j)^{2n} = -1.
$$

then $$(s/j)^{2n} = -1 = e^{j(2k-1)n}$$

where $k = 1, 2, \text{etc.}$

Hence $s^{2n} = e^{j(2k + n - 1)n}$

or $s_k = \exp j\left((2k + n - 1)/2n\right)\pi$

The poles therefore become $s_1, s_2 \ldots s_m$

and $G(s) = \frac{1}{(s + s_1)(s + s_2)\ldots}$

The transfer function for a 2nd order filter

Taking a simple example of $n = 2$ from equation (4)

$$s^4 = -1 = e^{j\pi}
$$

$s = \pm e^{j\pi/4}$

The result implies that the poles which occur in the left half plane are:

$$G(s) = \frac{1}{(s + e^{j\pi/4})(s + e^{-j\pi/4})}$$

$$= \frac{1}{(s^2 + \sqrt{2}s + 1)}
$$

Similar expressions can be derived for any order of filter.

The design of a 4th order Butterworth filter in the s-domain

The basic theory of the Butterworth approximation was developed in the previous section and illustrated by an example of the derivation of a 2nd order filter. The design of a 4th order low-pass Butterworth filter will now be considered. Referring to Figure 1, it is clear that the 4th order filter will break at approximately 0.8 rad sec$^{-1}$. From equation (2) the frequency response is given by:

$$|G(j\omega)|^2 = \frac{1}{1 + \omega^2}
$$

To obtain the transfer function of the filter equation (6) becomes:

$$G(s) G(-s) = \frac{1}{1 + s^2}
$$

The poles of the function are therefore the roots of

$$s^8 = -1 = e^{j\pi m}$$

$m = 1, 3, 5 \ldots$

i.e. $s = e^{j\pi/8}, e^{j3\pi/8}, e^{j5\pi/8}, e^{j2\pi/8}, e^{j7\pi/8}, e^{j9\pi/8}, e^{j11\pi/8}$

It can be seen from this result that all the poles lie on a unit circle in the s-plane (Figure 2 illustrates the equivalent result for 2nd and 3rd order filters).

The poles are evaluated by applying Euler's formula. Take for example $s_3$:

$$s_3 = e^{j5\pi/8} = \cos 5\pi/8 + j \sin 5\pi/8
$$

The same procedure can be repeated for all the poles in the left half of the s-plane and the transfer function becomes:

$$G(s) = \frac{1}{(s^2 - 2\cos 5\pi/8 + 1)(s^2 - 2\cos 7\pi/8 + 1)}
$$

This basic transfer function can be converted to its equivalent polynomial form as follows:

$$G(s) = \frac{1}{1 + 2.613s + 3.414s^2 + 2.613s^3 + s^4}
$$

This result leads to a more general statement relating to pole positions on the unit circle of the s-plane. For a given order of filter $n$, 2n poles are equally spaced on the unit circle. The poles are symmetrically located with respect to the imaginary axis. A pole is never located on the imaginary axis although for odd values of $n$, poles occur on the real axis. The angular spacing of poles is given by $\pi/n$.

Consider the example of a third order filter i.e. $n = 3$. The separation, $\theta$, is $\pi/3 = 60^\circ$.

Hence there will be 6 poles on the unit circle with pole pairs equally spaced about the imaginary axis. Figure 2b illustrates the distribution. For the filter realization only the poles which exist in the stable left half s-plane are used. From Figure 2b locations of those poles are:

$$s_1 = e^{j2\pi/3}
$$

$$s_2 = e^{j2\pi} = -1
$$

$$s_3 = e^{j4\pi/3}
$$
Therefore $G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$

The transfer function for any other order of Butterworth filter can be developed using the methods derived in this section. Table 1 lists the $s$-domain transfer functions for low-pass Butterworth filters of order $n = 1$ to 4.

The realization of Butterworth filters by electronic circuits, typically by means of operational amplifiers, requires the transfer function in Laplace operator form. The realization of a particular filter in digital form involves some additional steps. Digital filters operate on sampled data, hence the $s$-domain transfer function must be converted to the equivalent $z$-domain form before realization can be achieved. The final step in the digital realization procedure is to convert the $z$-transform transfer function into the equivalent time domain recurrence relationship.

In the example which will be considered, conversion to the $z$-domain is carried out by means of a bilinear transformation. As previously discussed in Part II, the bilinear transformation used for this purpose results in a nonlinear mapping of the $s$ to the $z$-plane. However for low-pass filters the mapping is approximately linear up to a quarter of the sampling frequency.

(a) A 2nd order low-pass filter. The transfer function of the 2nd order filter was previously derived and is listed in Table 1 as:

$$G(s) = \frac{1}{s^2 + \sqrt{2} s + 1}$$

The transfer function of equation (10) is for a filter normalized to a cut-off frequency of 1 radian per second. The filter can be generalized by replacing the Laplace operator $s$ by $s/c$, where $c$ is the cut-off frequency.

Therefore

$$G(s) = \frac{1}{(s/c)^2 + \sqrt{2} (s/c) + 1}$$

Substituting for $s$ via the bilinear transformation:

$$s = \frac{z - 1}{z + 1}$$

$$H(z) = \frac{c^2}{(z - 1)^2 + \sqrt{2} \frac{(z - 1)}{(z + 1)} c + c^2}$$

An important point to note at this stage of the derivation is that the frequency parameter $c$ becomes a function of the sampling frequency of the data. Hence in the $z$-domain the value of $c$ is given by:

$$c = \pi \omega_s / \omega_c$$
where \( \omega_c \) is the desired cut-off frequency and \( \omega_s \) the sampling frequency of the data.

Therefore

\[
y(z) = \frac{c^2(z^2 + 2z + 1)}{(z^2 - 2z + 1) + 1.414(z^2 - 1)c + c^2(z^2 + 2z + 1)}
\]

\[
= \frac{c^2z^2 + 2c^2z + c^2}{z^2(1 + 1.414c + c^2) + 2z(c^2 - 1) + (c^2 - 1.414c - 1)}
\]

(12)

Letting

\[
A = c^2
\]

\[
B = 1 + 1.414c + A
\]

\[
C = 2(c^2 - 1)
\]

\[
D = A - 1.414c + 1
\]

Therefore

\[
y(z) \left( z^2B + zC + D \right) = X(z) \left[ (A(z^2 + 2z + 1)) \right] \] (13)

The \( N \) transform equation (13) can now be converted to the time domain equivalent by the \( z \)-transform shift theorem i.e. \( z^{-1} \) implies a delay of one sample period.

Hence, covering to the time domain equation (13) becomes:

\[
y(n + 2)B + y(n + 1)C + y(n)D
\]

\[
= A[x(n + 2) + 2x(n + 1) + x(n)]
\]

(14)

and shifting both sides of the equation backwards two samples to make the filter realizable

\[
y(n) - \left( A[x(n) + 2x(n - 1) + x(n - 2)] \right)
\]

\[
- \left[ (C + D)[y(n - 1) + Dy(n - 2))] \right] / B
\]

(15)

The impulse and frequency responses of this filter are shown on Figure 3.

As an example of the use of the filter, the recurrence relationship of equation (15) was applied to ECG data which had been sampled at 100 Hz. The waveform used for the test is illustrated in Figure 3.

Figures 3 and 4 illustrate the results obtained when the filter (as defined by the algorithm above), is applied to an ECG waveform.

Figure 4 shows the time waveform and frequency responses for the 2nd order filter set for cut-off frequencies of 5 Hz. It can be seen from the figures that the frequency responses are as predicted from the theory with the half power points occurring at the defined cut-off frequencies.
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(b) Design of a 4th Order Butterworth Filter. As a second example of the digital realization of Butterworth filters we will consider the design of a 4th order filter. The basic design procedure is the same as for the previous example.

From Table 1, the transfer function of the filter in the 's' plane is:

$$G(s) = \frac{1}{1 + 2.613s + 3.414s^2 + 2.613s^3 + s^4} \quad (16)$$

If the cut-off frequency is again defined as $c - \text{rad sec}^{-1}$.

Then

$$G(s) = \frac{1}{c^4 + 2.613c^3s + 3.414c^2s^2 + 2.613cs^3 + s^4} \quad (17)$$

Now letting $A = c^4$, $C = 3.414c^2$, $B = 2.613c^3$, $D = 2.613c$

The $z$ transform transfer function is obtained by substituting

$$s = \frac{z - 1}{z + 1} \quad \text{in equation (17)}$$

So that equation (17) becomes

$$H(z) = \frac{A}{A + B\left[\frac{z - 1}{z + 1}\right] + C\left[\frac{z - 1}{z + 1}\right]^2 + D\left[\frac{z - 1}{z + 1}\right]^3 + \left[\frac{z - 1}{z + 1}\right]^4} \quad (18)$$

Multiplying numerator and denominator by $(z + 1)^4$ we arrive at a new expression for $H(z)$ whose numerator is

$$A(z^4 + 4z^3 + 6z^2 + 4z + 1)$$

and whose denominator is:

$$A(z^4 + 4z^3 + 6z^2 + 4z + 1)$$

$$+ B(z^4 + 2z^3 - 2z - 1)$$

$$+ C(z^4 - 2z^2 + 1)$$

$$+ D(z^4 - 2z^3 + 2z - 1)$$

$$+ (z^4 - 4z^3 + 6z^2 - 4z + 1)$$

Now collecting denominator terms
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in the case of the 4th order element very much less energy gets through the filter at frequencies greater than cut-off.

The recurrence relationships presented in equations (15) and (20-22) allow the implementation of 2nd and 4th order Butterworth filters which can be written in the form of a standard subroutine with the parameter c defining the cut-off frequency.

\[
ZP4 = A + B + C + D + 1 (z^4)
\]
\[
ZP3 = 4A + 2B + 2D - 4 (z^3)
\]
\[
ZP2 = 6A - 2C + 6 (z^2)
\]
\[
ZP1 = 4A - 2B + 2D - 4 (z)
\]

\[
CON = A - B + C - D + 1
\]

Now if \( H(z) = Y(z)/X(z) \)

\[
\frac{1}{H(z)} = \frac{ZP4.z^4 + ZP3.z^3 + ZP2.z^2 + ZP1.z + CON}{A(z^4 + 4z^3 + 6z^2 + 4z + 1)}
\]

and converting to the time domain by means of the z transform shift theorem gives:

\[
ZP4 y(n + 4) + ZP3 y(n + 3) + ZP2 y(n + 2)
+ ZP1 y(n + 1) + CON y(n) = A \left[ x(n + 4) + 4x(n + 3) + 6x(n + 2) + 4x(n + 1) + x(n) \right]
\]

(19)

If all the terms of equation (19) are delayed by 4 sample intervals, then:

\[
y(n) = (XTERM - YTERM)/ZP4
\]

(20)

where

\[
XTERM = A \left[ x(n) + 4x(n - 1) + 6x(n - 2)
+ 4x(n - 3) + x(n - 4) \right]
\]

and

\[
YTERM = ZP3 y(n - 1) + ZP2 y(n - 2)
+ ZP1 y(n - 3) + CON y(n - 4)
\]

(22)

Figure 5 shows the impulse and frequency responses of this filter when set to various cut-off frequencies.

Example of the 4th Order Filter

Figure 4 shows the effect of a 2nd order filter with a cut-off frequency of 5 Hz when applied to an ECG signal. Figure 6 shows the same data after being filtered by a 4th order element. It can be seen from the power spectra of the filtered signals that

\[
\left| G(j\omega) \right|^2 = \frac{1}{1 + \epsilon^2 C_n^2 (\omega)}
\]

(23)

As previously discussed, the basic design criterion for low pass filters is that the magnitude of the frequency response should be close to unity in the pass band. The Chebychev approximation is slightly more complicated than that developed by Butterworth, the form of the normalized frequency response being:

\[
\left| G(j\omega) \right|^2 = \frac{1}{1 + \epsilon^2 C_n^2 (\omega)}
\]

(23)

where \( \epsilon \) is a parameter which controls ripple in the pass band (\( \epsilon < 1 \)) and \( C_n \) is a Chebychev polynomial; the exact nature of these polynomials will be considered later. In terms of the response, the principal difference between the Chebychev and Butterworth approximations is that the Chebychev approximation trades a higher rate of cut-off in the stop band for a controlled degree of ripple in the pass band.

The Chebychev polynomials of equation (23) are given by:

\[
C_n (\omega) = \cos (n \cos^{-1} \omega) \quad 0 \leq \omega \leq 1
\]

(24)

and

\[
C_n (\omega) = \cosh (n \cosh^{-1} \omega) \quad \omega > 1.
\]

(25)

A recurrence relationship for generating the Chebychev polynomials can be derived from equation (24) by substituting standard trigonometrical identities viz:

\[
C_{n+1} (\omega) - 2 \omega C_n (\omega) + C_{n-1} (\omega) = 0
\]

with \( C_1 (\omega) = \omega \) and \( C_2 (\omega) = 2\omega^2 - 1 \)

Table 2 gives the Chebychev polynomials derived from the relationship, up to the 8th order.
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Table 2

<table>
<thead>
<tr>
<th>n</th>
<th>(C_n(\omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\omega)</td>
</tr>
<tr>
<td>2</td>
<td>(2\omega^3 - 1)</td>
</tr>
<tr>
<td>3</td>
<td>(4\omega^3 - 3\omega)</td>
</tr>
<tr>
<td>4</td>
<td>(8\omega^5 - 8\omega^3 + 1)</td>
</tr>
<tr>
<td>5</td>
<td>(16\omega^5 - 20\omega^3 + 5\omega)</td>
</tr>
<tr>
<td>6</td>
<td>(32\omega^6 - 48\omega^4 + 18\omega^2 - 1)</td>
</tr>
<tr>
<td>7</td>
<td>(64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega)</td>
</tr>
<tr>
<td>8</td>
<td>(128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1)</td>
</tr>
</tbody>
</table>

As can be seen from equation 23 the ripples in the passband lie between unity and \(\frac{1}{(1 + \epsilon^2)^{\frac{n}{2}}}\).

And for \(\omega \gg 1\): \(G(j\omega) = \frac{1}{\epsilon C_n(\omega)}\) from equation 23.

Taking logs:

\[
20 \log_{10} G(j\omega) = -20 \log_{10} \epsilon - 20 \log_{10} C_n(\omega)
\]

But for \(\omega > 1\): \(C_n(\omega) = 2^{-n-1} (\omega^n)\)

Substituting in the right hand side of the above equation gives the loss in decibels.

\[
\text{loss} = 20 \log_{10} \epsilon + 6.02 (n - 1) + 20n \log_{10} \omega
\]

At high frequencies the fall off is the standard \(6n\) db/octave.

**Determination of the poles of the filter**

The poles of the filter can be determined in a manner similar to that used for the Butterworth filter, that is, by calculating the roots of the denominator of equation (18).

If \(s = j\omega\) then \(\omega = s/j\)

\[
1 + \epsilon^2 C_n^2 (s/j) = 0
\]

or

\[
C_n (s/j) = \cos (n\alpha + nj\beta) = \pm j 1/\epsilon
\]

or

\[
\cos \alpha \cosh \beta - jsin \alpha \sinh \beta = \pm j 1/\epsilon
\]

Therefore equating real and imaginary parts:

\[
\cos \alpha \cosh \beta = 0
\]

and

\[
\sin \alpha \sinh \beta = \pm 1/\epsilon
\]

But

\[
cosh \beta \neq 0 \text{ therefore } \cos \alpha = 0
\]

Therefore

\[
\alpha = \pm \frac{\pi}{2n}, \pm \frac{3\pi}{2n}, \pm \frac{5\pi}{2n}, \ldots
\]

Hence

\[
\sin n \alpha = \pm 1 \text{ and consequently}
\]

\[
\sinh n \beta = 1/\epsilon \text{ therefore } \beta = 1/n \sinh^{-1} 1/\epsilon
\]

As \(\alpha\) and \(\beta\) are now defined, substituting back in equation (30)

\[
s = j \cos (\alpha + j\beta)
\]

or in its alternative form

\[
s = j [\cos \alpha \cosh \beta - jsin \alpha \sinh \beta]
\]

or

\[
s = \sin \alpha \sinh \beta + j\cos \alpha \cosh \beta
\]

This is an important result because equation (39) can be used to determine the poles of a particular filter.

**Design of a 3rd Order Chebychev Filter**

Let us now consider the application of equation (39) to the design of a 3rd order lowpass filter with unity gain in the passband and maximum allowable ripple of 0.5db. The steps in the digital realization are identical to those previously used in the examples of Butterworth realization, i.e. design in the \(s\)-domain, realization in the \(z\)-domain and finally, conversion to the time domain.

**Design in the \(s\)-domain**

The first step in the \(s\)-domain calculations is the determination of a value for the parameter \(\epsilon\) (equation 23) which meets the design criterion of 0.5db ripple.

\[
\text{Hence } 20 \log_{10} \sqrt{1 + \epsilon^2} \leq 0.5
\]

Therefore \(\epsilon \leq 0.349\).
Hence from equation (37) in order to determine the poles of the filter from equation (39) the values of \(\alpha\) and \(\beta\) must first be determined.

\[
\beta = \frac{1}{n} \sinh^{-1} \frac{1}{e}
\]

Therefore

\[
\beta = \frac{1}{3} \sinh^{-1} \frac{1}{0.349} = \frac{1}{3} \cdot 1.775 = 0.5916
\]

Also from equation (37) \(\sin n \alpha = \pm 1\).

\[
\alpha = \pm \frac{\pi}{2n}, \pm \frac{3\pi}{2n}, = \pm \frac{\pi}{6}, \pm \frac{3\pi}{6}, \pm \frac{5\pi}{6}
\]

Substituting these values in equation (39) gives the values of the poles of the filter.

Hence

\[
s_1 = \sin \frac{\pi}{6} \sinh 0.5916 + j \cos \frac{\pi}{6} \cosh 0.5916 = 0.3124 + j 1.02
\]

\[
s_2 = 0.3124 - j 1.02 \text{ (conjugate pole)}
\]

and

\[
s_3 = \sin \frac{\pi}{2} \sinh 0.5916 = 0.6248
\]

Therefore the transfer function of the filter \(G(s)\) is given by:

\[
G(s) = \frac{\text{Constant}}{(s + 0.3124 + j 1.02)(s + 0.3124 - j 1.02)(s + 0.6248)}
\]

\[
= \frac{0.711}{s^3 + 1.25s^2 + 1.53s + 0.711}
\]

Correct to three significant figures and adjusted for unity gain in the passband. (The numerator constant is determined by setting \(s = j\omega\) in equation (41) and setting the constant for unity gain at zero frequency).

Realization in the \(z\) domain

Realization of the filter in the \(z\) domain requires (a) that the cut-off frequency is generalized by replacing \(s\) in equation (41) by \(s/c\) and (b) using the bilinear transformation, \(s = -1/z + 1\) to redefine equation (41)

Hence

\[
H(z) = \frac{A}{A + B \left(\frac{z - 1}{z + 1}\right) + C \left(\frac{z - 1}{z + 1}\right)^2 + D \left(\frac{z - 1}{z + 1}\right)^3}
\]

where \(A = 0.711c^3, B = 1.53c^2, C = 1.25c, D = 1.0\) and the denominator simplifies to four terms.

\(c = \pi \omega_c / \omega_s\) where \(\omega_c\) = cut off frequency and \(\omega_s\) = the sampling frequency.

\[
A(z^3 + 3z^2 + 3z + 1)
\]

\[
B(z^3 + z^2 - z - 1)
\]

\[
C(z^3 - z^2 - z + 1)
\]

\[
D(z^3 - 3z^2 + 3z + 1)
\]

Collecting powers of \(z\)

\[
ZP3 = A + B + C + D \quad (z^3)
\]

\[
ZP2 = 3A + B - C + 3D \quad (z^2)
\]

\[
ZP1 = 3A - B - C + 3D \quad (z)
\]

\[
CON = A - B + C - D \quad \text{(constant)}
\]

Substituting back into equation (42)

\[
H(z) \text{ becomes}
\]

\[
H(z) = \frac{A(z^3 + 3z^2 + 3z + 1)}{ZP3z^3 + ZP2z^2 + ZP1z + CON}
\]

Conversion to the Time Domain

The \(z\) domain transfer function of the filter can be converted to an equivalent time domain recurrence relationship by employing the same technique as was used in the previous examples i.e. the \(z\) transform shift theorem.

Hence the equivalent recurrence relationship in the time domain is:

\[
y(n + 3) . ZP3 + y(n + 2) . ZP2 + y(n + 1) . ZP1 + CON . y(n) = A \left[x(n + 3) + 3x(n + 2) + 3x(n + 1) + x(n)\right]
\]

and shifting for the \(y(n)^{th}\) sample

\[
XTERM = A x(n) + 3x(n - 1) + 3x(n - 2) + 3x(n - 3)
\]

and

\[
YTERM = ZP2 y(n - 1) + ZP1 . y(n - 2) + CON . y(n - 3)
\]

Then

\[
y(n) = \frac{(XTERM - YTERM)}{ZP3}
\]

Application of the filter to ECG data

In order to test the design, the impulse and frequency responses were calculated as before, and then the filter was applied to the same set of ECG data which we employed previously to test the Butterworth designs.

Figure 7 shows the impulse and frequency responses of the filter; it can be seen that the half power points occur at the designed frequencies. It can also be seen from the figure that the rate of fall-off of the
Chebychev filter is greater than for the equivalent Butterworth but this is achieved at the expense of the shape of the characteristic in the passband. The effect of the filter on our ECG data is shown on Figure 8.

Graphical construction of the Chebychev s plane poles
In the earlier section on the design of a Butterworth filter it was shown that the poles of the filter are equally spaced on the unit circle in the s plane, the poles of the Chebychev filter in contrast, lie on an ellipse in the s plane. As an alternative to algebraic design a graphical method can be employed to determine the s plane pole positions for low pass Chebychev filters. The basis of the method is the definition of two circles on the s plane, the diameters of which correspond to the lengths of the major and minor axes of the Chebychev ellipse.

The minor axis length $a$ is defined as:

$$a = (\gamma^{1/n} - \gamma^{-1/n})$$  \hspace{1cm} (47)

where $\gamma = e^{-1} + \sqrt{1 + e^{-2}}$.

The major axis is similarly defined by:

$$b = (\gamma^{1/n} + \gamma^{-1/n})$$

The poles of the Chebychev filter are obtained as follows:

Step 1. Identify the points on the major and minor circles equally spaced by $\pi/n$.
Step 2. Ensure that the points are symmetrically spaced with respect to the imaginary axis.

Step 3. Ensure that a point never falls on the imaginary axis and that a point occurs on the real axis for \( n \) odd.

Example

Let us now reconsider the design of the 3rd Order filter discussed in an earlier section.

\( e \) was found to be

\[ e = 0.349 \quad \text{and} \quad n = 3. \]

The first step in the design procedure is to locate the points on the major and minor axes. In this case they are

\[ n/3 \text{ apart i.e. } \pi/3. \]

The lengths of the minor and major axes are calculated as follows: From equation (47)

\[ \gamma = e^{-1} + \sqrt{1 + e^{-2}} \]

\[ = \frac{1}{0.349} + \sqrt{1 + \frac{1}{(0.349)^2}} = 5.89 \]

Substituting for \( \gamma \) in equation (47)

\[ a = \frac{1}{2}(\gamma^{1/3} - \gamma^{-1/3}) \]

hence

\[ a = \frac{5.89^{1/3}}{2} - \frac{1}{5.89^{1/3}} = 0.62 \]

and

\[ b = \frac{1}{2}(\gamma^{1/3} + \gamma^{-1/3}) \]

\[ = 1.175 \]

Plotting the minor and major circles on the \( s \) plane (Figure 9) and drawing the lines defined by \( n/3 \) give the basic construction for defining the pole positions.

The intersection of these lines with the minor circle locates points on the vertical pole location line. The intersection of the \( n/3 \) lines with the major circle defined points on the horizontal pole location line. Hence the positions of the four poles can be found. In this case their values are:

\[ s_1 = 0.31 + j 1.0 \]

\[ s_2 = 0.31 - j 1.0 \]

\[ s_{3,4} = \pm 0.62 \]

which are approximately the same values as were found algebraically.

Although the graphical methods for obtaining the location of \( s \) plane poles is rapid and convenient, it is important to remember that the technique will only give an approximate answer. This, nevertheless,
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It is possible to derive convenient recurrence relationships for digital filters which enhance their speed of operation. The use of the z plane for real filter design rather than translation from the s plane is of critical importance.

In the second part of the paper the idea of using the z plane in terms of a transfer function relating input and output was introduced. Design of filters in the z plane has the major advantage that translation back into the time domain is an easy matter, and to this end the relationship between simple pole configurations in the s domain and the z domain has been considered. Transformation from the s domain to the z domain can be achieved by a number of techniques. However, perhaps the most commonly used approach is via the bilinear transformation. In its most common form the transformation allows the whole jω axis from the s plane to be mapped onto one revolution of the unit circle in the z plane. The right half s plane maps outside the unit circle on the z plane and the left half s plane maps inside it.

As illustrated in Part III, the design of more complex digital filters, for example, the Butterworth and Chebychev, even by means of the bilinear transformation, is a somewhat long algebraic procedure. To overcome this difficulty the Smith chart is often used. This chart is a graphical representation of the bilinear transformation. Hence poles plotted on the normal s domain can be read off the Smith chart in their z domain equivalent form. The disadvantage of this method is that the accuracy with which the poles can be located is limited. Nevertheless, the technique is often used with success and can always provide a very useful check.

The discussion on the design of different types of Butterworth and Chebychev filters has led to the development of a suite of digital filter algorithms. Although the algebra is somewhat complex, once these filters have been designed they can be generalized by setting up a single parameter which represents their cut-off frequency. Although such filters are very efficient in terms of providing a characteristic which approximates to the ideal case without significant side lobes, and with flat pass bands, there major disadvantage is the speed with which the calculation can be achieved. Hence they find their most useful implementation in operations which do not require fast on-line processing and where optimised frequency responses are required.

In conclusion, we should emphasise the fact that digital filters must be chosen carefully to match the tasks which are required of them. Obviously simple designs should be favoured if their frequency responses are adequate and care should be taken to create filtering algorithms which run most efficiently on the computer system in question. Before a filter is applied to real data it should be tested in a program which yields its impulse and frequency responses so that its applicability can be confirmed.

REFERENCES